4AE-SE - 4FISA
BE - Commande Numerique
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[BE] Digital control of a cantilever beam

Rules of the report

- **6 pages maximum. No need to copy the BE text neither the questions text.**
- **You can use Matlab, Octave, Python, Mathcad, or other softwares for simulation and complex calculations.**
- **No type of Artificial Intelligence is allowed. Only human intelligence is permitted.**
- **Precise the answers of the questions. Example "Answer of Question 1 : ..."**
- **Precise the names and last names of the group.**
- **The deadline is 08/05/2026. You have plenty of time, more than 2 months.**
- **The groups are given in Moddle.**

1 Modelling

1.1 Formulation of the problem

The cantilever beam is usually used to model structures like bridges, towers, building, aircraft wings, microelectromechanical systems (MEMS), among others. In this Bureau d'Étude (BE), we consider a clamped beam at the left side with a force actuator at the right side. The elastic behavior of the beam makes challenging its control. In a first instance, the objective of this BE is the control of the oscillations and tracking a desired end-tip position. In a second instance, we aim to analyze the possible implementation with a digital controller, specifying the possible sampling period that guarantees the stability of the system. In the following two animations, one can observer the open loop behavior and the desired closed-loop response that tracks a desired end-tip position.

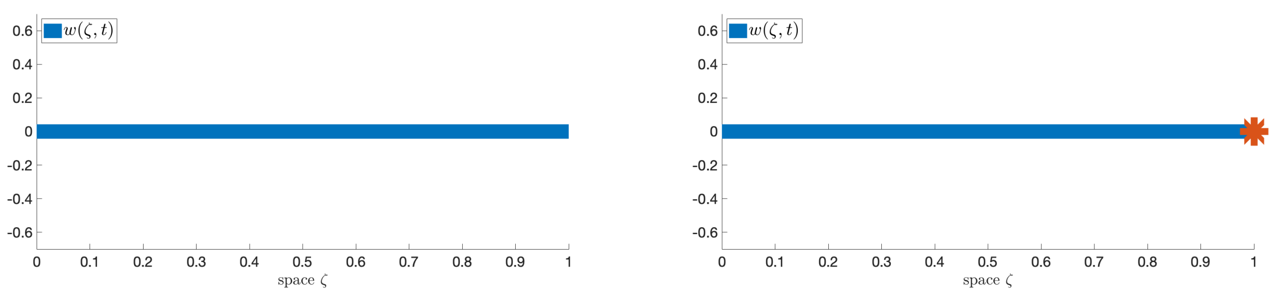

Figure 1 – *Left Animation.* Cantilever beam with a constant force applied at the left side. *Right Animation.* Cantilever beam in closed-loop with a tracking of the end-tip position.

1.2 Dynamics using Partial Differential Equations (PDEs)

We consider the following PDE model :

$$\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) = -\frac{\partial^2}{\partial \zeta^2}\left(EI(\zeta)\frac{\partial^2 w}{\partial \zeta^2}(\zeta,t)\right) - q(\zeta,t) \tag{1}$$

where $\zeta \in [0, L]$ is the spatial independent variable and $t \geq 0$ is the time independent variable. The length of the beam is $L = 1\,[m]$. $\rho(\zeta)$ is the mass density (mass per unit length), E is the elastic modulus and $I(\zeta)$ is the second moment of area. $w(\zeta,t)$ is the beam deformation and $q(\zeta,t)$ is an external load on the beam.

— $\dfrac{\partial w}{\partial t}(\zeta, t)$ is the velocity,

— $\dfrac{\partial w}{\partial \zeta}(\zeta, t)$ represents the slope of the beam at each spatial point,

— $EI(\zeta, t)\dfrac{\partial^2 w}{\partial \zeta^2}(\zeta, t)$ is the bending moment, and

— $\dfrac{\partial}{\partial \zeta}\left(EI(\zeta, t)\dfrac{\partial^2 w}{\partial \zeta^2}(\zeta, t)\right)$ is the shear force.

We define two state variables

$$\begin{aligned} x_1(\zeta, t) &= \frac{\partial^2 w}{\partial \zeta^2}(\zeta, t), \\ x_2(\zeta, t) &= \rho(\zeta)\frac{\partial w}{\partial t}(\zeta, t). \end{aligned} \tag{2}$$

Additionally, we define two auxiliary variables, called effort variables as :

$$\begin{aligned} e_1(\zeta, t) &= EI(\zeta)x_1(\zeta, t), && \text{Bending moment,} \\ e_2(\zeta, t) &= \frac{1}{\rho(\zeta)}x_2(\zeta, t), && \text{Velocity.} \end{aligned} \tag{3}$$

Question 1 *Find the state space representation on the following form $\dot{x} = f(e_1, e_2, q)$.*

1.3 Boundary Conditions

We consider that the beam is clamped at the left side, *i.e.,* at $\zeta = 0$. This implies the following conditions on the left side of the beam :

$$\begin{aligned} e_2(0, t) &= 0, && \text{Velocity at the left equal to zero} \\ \left.\frac{\partial e_2}{\partial \zeta}(\zeta, t)\right|_{\zeta=0} &= 0, && \text{Angular velocity at the left equal to zero} \end{aligned} \tag{4}$$

At the right side, we consider that we have a force actuator that can apply a force in the vertical axis and we consider no torque at the right side.

$$\begin{aligned} \left.\frac{\partial e_1}{\partial \zeta}(\zeta, t)\right|_{\zeta=L} &= u(t), && \text{Force applied at the right side} \\ e_1(L, t) &= 0, && \text{Torque applied at the right side.} \end{aligned} \tag{5}$$

Finally, we consider that we can measure the velocity at the right side of the beam

$$y(t) = -e_2(L, t). \tag{6}$$

1.4 Finite Element Discretization

The states x_1, and x_2, and the effort variables e_1 and e_2 are approximated as follows :

$$\begin{aligned} x_1(\zeta, t) &\approx \phi(\zeta)^\top x_{1d}(t), \\ x_2(\zeta, t) &\approx \phi(\zeta)^\top x_{2d}(t), \\ e_1(\zeta, t) &\approx \phi(\zeta)^\top e_{1d}(t), \\ e_2(\zeta, t) &\approx \phi(\zeta)^\top e_{2d}(t), \end{aligned} \quad \text{with} \quad \phi(\zeta) = \begin{bmatrix} 2\zeta^3 - 3\zeta^2 + 1 \\ 3\zeta^2 - 2\zeta^3 \\ \zeta^3 - 2\zeta^2 + \zeta \\ \zeta^3 - \zeta^2 \end{bmatrix}, \quad \text{and } x_{1d},\, x_{2d},\, e_{1d}, e_{2d} \in \mathbb{R}^{4\times 1}. \tag{7}$$

Question 2 *Show that the state space equations are given by*

$$\begin{aligned} E\dot{x}_{1d}(t) &= De_{2d}(t), \\ E\dot{x}_{2d}(t) &= -D^\top e_{1d}(t) - \phi(L)u(t) - F_{ext}, \\ y(t) &= -\phi(L)^\top e_{2d}. \end{aligned} \tag{8}$$

with

$$E = \int_0^L \phi(\zeta)\phi(\zeta)^\top d\zeta = \begin{bmatrix} 13/35 & 9/70 & 11/210 & -13/420 \\ 9/70 & 13/35 & 13/420 & -11/210 \\ 11/210 & 13/420 & 1/105 & -1/140 \\ -13/420 & -11/210 & -1/140 & 1/105 \end{bmatrix}, \qquad F_{ext} = \int_0^L \phi(\zeta)q(\zeta,t)d\zeta, \tag{9}$$

$$D = \phi(L)\frac{\partial \phi}{\partial \zeta}(L)^\top - \frac{\partial \phi}{\partial \zeta}(L)\phi(L)^\top + \int_0^L \frac{\partial^2 \phi}{\partial \zeta^2}(\zeta)\phi(\zeta)^\top d\zeta = \begin{bmatrix} -6/5 & 6/5 & -1/10 & -1/10 \\ 6/5 & -6/5 & 1/10 & 11/10 \\ -11/10 & 1/10 & -2/15 & 1/30 \\ -1/10 & 1/10 & 1/30 & -2/15 \end{bmatrix} \tag{10}$$

Question 3 *Consider unitary and constant parameters rho $= EI = 1$ and zero external load $q(\zeta,t) = 0$. Provide the numerical values of the matrices A, B and C of the following state space representation*

$$\Sigma \begin{cases} \dot{x}_d(t) = Ax_d(t) + Bu(t), \\ y(t) = Cx_d(t). \end{cases} \tag{11}$$

with $x_d(t) = \begin{bmatrix} x_{1d}(t) \\ x_{2d}(t) \end{bmatrix}$.

Question 4 *We want to visualize the eigenvalues after time discretization for three different sampling periods $T_{s1} = 0.01$, $T_{s2} = 0.02$ and $T_{s3} = 0.04$. In the same figure, provide 3 subplots of the complex plane with (i) the continuos-time eigenvalues, (ii) the discrete-time eigenvalues using the Tustin method and (iii) the discrete-time eigenvalues using the zero-order hold. What can we conclude about stability for all the cases.*

Question 5 *Numerical simulation of the open-loop continuous-time system. Consider the simulation time $t \in [0, 10]$, null initial conditions and unitary step at second $t = 1$ as input signal. Provide a figure with the input signal $u(t)$ and the output signal $y(t)$.*

Question 6 *Show that we can reconstruct the beam deformation $w(\zeta,t)$ as*

$$w(\zeta,t) = C_w(\zeta)x_d(t), \tag{12}$$

with

$$C_w(\zeta) = \begin{bmatrix} \frac{\zeta^2(2\zeta^3 - 5\zeta^2 + 10)}{20} & \frac{-\zeta^4(2\zeta - 5)}{20} & \frac{\zeta^3(3\zeta^2 - 10\zeta + 10)}{60} & \frac{\zeta^4(3\zeta - 5)}{60} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

1.5 Animation

You can use the code *Animation.m* to show the beam deformation. This function has as input x_d and t, with x_d being the simulated state and t the time simulation vector. If for example, you have N_t samples, the sizes must be $x_d \in \mathbb{R}^{8\times N_t}$ and $t \in \mathbb{R}^{1\times N_t}$. For correct visualization of the animation use a time step $\delta_t = 0.0001\,[s]$.

2 Output Feedback

We are interested to apply an output feedback control with a pre-gain action as follows :

$$u(t) = -ky(t) + Hw_c(L,t). \tag{14}$$

with $k > 0$, $H \in \mathbb{R}$, and $w_c(L,t)$ the desired (consigne) value of the end-tip position of the beam. We want that $w(L,t)$ converges to $w_c(L,t)$ in steady state.

Question 7 *By test and error, find the k that gives the fastest settling time and find the H that guarantees the tracking of the signal $w(L,t)$ to the desired value $w_c(L,t)$*

Question 8 *Numerical simulation of the closed-loop continuous-time system. Consider the simulation time $t \in [0, 10]$, null initial conditions and unitary step at second $t = 1$ as reference signal. Provide a figure with the input signal $u(t)$, the output signal $y(t)$, the reference of the end-tip position $w_c(L,t)$ and the end-tip position $w(L,t)$.*

2.1 Zero-order hold for the output feedback

We aim now to analyze the stability robustness with respect to implementation of this control law using a digital controller and a digital-to-analog converter. We assume that the control signal si constant between sampling time, so the continuous-time model is precede by a zero-order hold.

Question 9 *Consider the same three examples of time-discretization with a zero-order hold than in Question 4. In the same figure, provide the following 2 subplots : (i) closed-loop continuous-time eigenvalues, and (ii) closed-loop discrete-time eigenvalues using the zero-order hold for the three proposed sampling periods. What can you conclude with respect to the proposed sampling periods ?*

Question 10 *Considering a maximum of three decimal, what is the maximum sampling period that we can use to guarantee asymptotic stability ? Do a numerical simulation with $t \in [0, 10]$, null initial conditions and unitary step at second $t = 1$ as reference. Provide a figure with the input signal $u(t)$, the output signal $y(t)$, the reference of the end-tip position $w_c(L, t)$ and the end-tip position $w(L, t)$.*

3 State Feedback

We consider now a state feedback plus pre-grain control law of the form

$$u(t) = -Kx(t) + Hw_c(L, t). \tag{15}$$

with $K \in \mathbb{R}^{1\times 8}$, $H \in \mathbb{R}$, and $w_c(L, t)$ the desired (consigne) value of the end tip position of the beam. We want that $w(L, t)$ converges to $w_c(L, t)$

Question 11 *Design the matrix K, such that the following cost function is minimized :*

$$J = \int_0^\infty x_d(t)^\top x_d(t) + u(t)^2 dt \tag{16}$$

Provide the value of K (use only two decimals for each value).

Question 12 *Numerical simulation of the closed-loop continuous-time system. Consider the simulation time $t \in [0, 10]$, null initial conditions and unitary step at second $t = 1$ as reference signal. Provide a figure with the input signal $u(t)$, the output signal $y(t)$, the reference of the end-tip position $w_c(L, t)$ and the end-tip position $w(L, t)$.*

3.1 Zero-order hold for the state feedback

Similarly to the previous section, we aim to analyze the stability when implementing the state feedback control law in a digital controller.

Question 13 *Is the same sampling period provided in the previous section appropriate for this control law ? If not, provide the maximal value (up to 4 decimals) of the sampling period that guarantees asymptotic stability. Provide the numerical simulation as well.*

Question 14 *Propose a realizable sampling period in such a way that we can approach the desired behavior obtained in Question 12. Provide the numerical simulation as well.*

4 Perturbation rejection

We consider now an external distributed force $q(\zeta, t) = q_0(t)$ in the model (1). This means that the F_{ext} of the finite element model (8) is different to zero. The dynamic model (11) obtained in Question 3 is slightly modified.

Question 15 *Find the matrix B_p related to the perturbation value of $q_0(t)$ for the modified dynamic model*

$$\Sigma \begin{cases} \dot{x}_d(t) = Ax_d(t) + Bu(t) + B_p q_0(t), \\ y(t) = Cx_d(t). \end{cases} \tag{17}$$

Question 16 *Numerical Simulation. Provide the same simulation as in the previous questions, but now with the perturbation signal. Consider a step function at second $t = 4\,[s]$ with magnitude -5 for the perturbation signal $q_0(t)$ Can we still guarantee zero error in steady state ?*

4.1 Replacing the pre-gain H by an integral action

We are going to design an state feedback control law with an integral action. In the continuous-time model, the control law has the following form :

$$u(t) = -K_1 x(t) - K_i x_i(t) \tag{18}$$

with

$$\dot{x}_i(t) = w_c(L,t) - w(L,t). \tag{19}$$

If we define the augmented state as $x_{aug}(t) = \begin{bmatrix} x(t) \\ x_i(t) \end{bmatrix}$, the augmented state space representation becomes

$$\dot{x}_{aug}(t) = A_{aug} x_{aug}(t) + B_{aug} u(t) + B_{paug} q_0(t) + B_{ref} w_c(L.t) \tag{20}$$

with

$$A_{aug} = \begin{bmatrix} A & 0_{8\times1} \\ -C_w(L) & 0 \end{bmatrix}, \quad B_{aug} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad B_{paug} = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \quad B_{ref} = \begin{bmatrix} 0_{8\times1} \\ 1 \end{bmatrix}$$

For the augmented system, we aim to design a state feedback $u(t) = -K_{aug} x_{aug}(t)$ using pole placement. We aim to retain the same dynamic obtained in Question 11.

Question 17 *Provide the desired eigenvalues and propose the extra eigenvalue (due to the integral action) in such a way that we respect the behavior of the control law.*

Question 18 *Provide the gains K_1 and K_i.*

Question 19 *Provide the same numerical simulation as in Question 16, but now using the integral action. Is the perturbation rejected ?*

Question 20 *How can this type of control law can be implemented in a digital controller. Can we use the same gains K_1 and K_i ? Which is the maximal value of the sampling period that guarantees stability ? (Consider 5 decimals) Propose a sampling period that guarantees closed-loop stability and the desired performance.*