BE COMMANDE NUMÉRIQUE

Wissal GUARNI
Justin BOS
Nolan REYNIER NOMER
Aleksander TABAN
Elèves Ingénieurs de l'INSA Toulouse
Département GEI
Spécialité AE-SE
Promotion 60
2022-2027

Digital control of a cantilever beam

Bureau d'études en Commande Numérique

Professeurs
Jesus-Pablo TOLEDO-ZUCCO
Yassine ARRIBA

SOMMAIRE

1 Modelisation **1**

1.1 Question 1 1
1.2 Question 2 1
1.3 Question 3 2
1.4 Question 4 3
1.5 Question 5 4
1.6 Question 6 4

2 Retour de sortie **5**

2.1 Question 7 5
2.2 Question 8 5
2.3 Question 9 6
2.4 Question 10 6

3 Retour d'état **6**

3.1 Question 11 6
3.2 Question 12 6
3.3 Question 13 7
3.4 Question 14 7

4 Rejection de perturbation **8**

4.1 Question 15 8
4.2 Question 16 8
4.3 Question 17 8
4.4 Question 18 9
4.5 Question 19 9
4.6 Question 20 9

5 Annexe **A**

1 MODELISATION

$$\dot{x_1}, \dot{x_2} = f(e_1, e_2, q) \qquad \begin{cases} x_1(\zeta, t) = \dfrac{\partial^2 \omega}{\partial \varepsilon^2} \\ x_2(\zeta, t) = \rho(\zeta)\dfrac{\partial \omega}{dt} \end{cases} \qquad \begin{cases} e_1(\zeta, t) = EI(\zeta)x_1 \\ e_2(\zeta, t) = \dfrac{1}{\rho(\zeta)}x_2 = \dfrac{\partial \omega}{\partial t} \end{cases}$$

1.1 Question 1

$$\begin{cases} \dot{x_1} = \dfrac{\partial x_1}{\partial t} = \dfrac{\partial}{\partial t}(\dfrac{\partial^2 \omega}{\partial \zeta^2}) = \dfrac{\partial^2}{\partial \zeta^2}(\dfrac{\partial^2 \omega}{\partial t}) \\ \dot{x_2} = \rho(\zeta)\dfrac{\partial^2 \omega}{\partial t} = \underbrace{-\dfrac{\partial^2}{\partial \zeta^2}(EI(\zeta)\dfrac{\partial^2 \omega}{\partial \zeta^2}) - q(\zeta, t)}_{\text{EDP}} \end{cases} \Rightarrow \boxed{\begin{cases} \dot{x_1} = \dfrac{\partial^2 e_2}{\partial \zeta^2} \\ \dot{x_2} = -\dfrac{\partial^2 e_1}{\partial \zeta^2} - q(\zeta, t) \end{cases}} \tag{1}$$

1.2 Question 2

$$\begin{cases} x_1 \approx \phi^T x_{1d}(t) \\ x_2 \approx \phi^T x_{2d}(t) \end{cases} \qquad \begin{cases} e_1 \approx \phi^T e_{1d}(t) \\ e_2 \approx \phi^T e_{2d}(t) \end{cases}$$

En utilisant la première ligne de l'équation 1, on trouve :

$$\int \phi(\zeta)d\zeta \times \phi^T \dot{x_{1d}} = \int \phi(\zeta)d\zeta \times \frac{\partial^2}{\partial \zeta^2} = \ddot{\phi}(\zeta)^T e_{2d}$$

$$\underbrace{\int_0^L \phi(\zeta)\phi(\zeta)^T d\zeta}_{\text{E}} \times \dot{x_{1d}} = \left(\int_0^L \phi(\zeta)\ddot{\phi}(\zeta)d\zeta\right) e_{2d} \tag{2}$$

On applique plusieurs fois de l'intégration par partie (IPP) :

$$\begin{aligned} \int_0^L \phi(\zeta)\ddot{\phi}(\zeta)^T d\zeta &= [\phi(\zeta)\dot{\phi}(\zeta)^T]_0^L - \int_0^L \dot{\phi}(\zeta)\dot{\phi}(\zeta)^T d\zeta \\ &= \phi(L)\dot{\phi}(L)^T - \underbrace{\phi(0)\dot{\phi}(0)^T}_{=0} - \dot{\phi}(L)\phi(L)^T + \underbrace{\dot{\phi(0)}\phi(0)^T}_{=0} + \int_0^L \ddot{\phi}(\zeta)\phi(\zeta)^T d\zeta \\ &= \phi(L)\dot{\phi}(L)^T - \dot{\phi}(L)\phi(L)^T + \int_0^L \ddot{\phi}(\zeta)\phi(\zeta)^T d\zeta = D \end{aligned}$$

On continue avec ça en remplaçant dans l'équation 2 : $\Rightarrow \boxed{E\dot{x}_{1d} = De_{2d}}$

Puis on a dans la deuxième ligne de l'équation 1 :

$$
\begin{aligned}
\phi^T(\zeta)\dot{x}_{2d} &= -\frac{\partial^2}{\partial\zeta^2}(\phi(\zeta)^T e_{1d}) - q(\zeta,t) \\
\int_0^L \phi^T(\zeta)d\zeta \times \dot{x}_{2d} &= -e_{1d}\int_0^L \phi(\zeta)\ddot{\phi}(\zeta)^T d\zeta - \underbrace{\int_0^L \phi(\zeta)q(\zeta,t)d\zeta}_{=F_{ext}}
\end{aligned}
$$

Puis on trouve D :

$$
\begin{aligned}
&D = \phi(L)\dot{\phi}(L)^T - \phi(L)\dot{\phi}(L)^T + \int_0^L \ddot{\phi}(\zeta)\phi(\zeta)^T d\zeta \\
\Rightarrow &D^T = \dot{\phi}(L)\phi(L)^T - \dot{\phi}(L)\phi(L)^T + \int_0^L \phi(\zeta)\ddot{\phi}(\zeta)d\zeta \\
\Rightarrow &\boxed{\int_0^L \underbrace{\phi(\zeta)}\ddot{\phi}(\zeta)^T d\zeta} = D^T - \dot{\phi}(L)\phi(L)^T + \phi(L)\dot{\phi}(L)^T \\
\Rightarrow &E\dot{x}_{2d} = -e_{1d}D^T + \overbrace{e_{1d}\dot{\phi}(L)\phi(L)^T}^{=0} - e_{1d}\phi(L)\dot{\phi}(L)^T - F_{ext}
\end{aligned}
$$

Or on sait que :

$$
\Rightarrow \begin{cases} \dfrac{de_1}{d\zeta} = u(t) \Rightarrow \dot{\phi}(L)^T e_{1d} = u(t) \\ e_1(L,t) = 0 \Rightarrow \phi(L)^T e_{1d} = 0 \end{cases} \qquad \Rightarrow \qquad \boxed{E\dot{x}_{2d} = -e_{1d}D^T - \phi(L)u(t) - F_{ext}}
$$

Puis on a ceci :

$$
\begin{cases} y(t) = -e_2(L,t) \\ e_2(L,t) \approx \phi(L)^T e_{2d} \end{cases} \qquad \Rightarrow \qquad \boxed{y(t) = -\phi(L)^T e_{2d}}
$$

1.3 Question 3

On a :

$$
\begin{cases} e_1 = EIx_1 = x_1 \Rightarrow e_{1d} = x_{1d} \\ e_2 = \dfrac{1}{\rho(\zeta)}x_2 = x_2 \Rightarrow e_{2d} = x_{2d} \end{cases}
$$

Donc :

$$
\begin{cases} E\dot{X}_{1d} = Dx_{2d} \\ E\dot{X}_{2d} = -D^T x_{1d} - \phi(L)u(t) - 0 \\ y = -\phi(L)^T x_{2d} \end{cases}
$$

Ce qui nous donne :

$$A = \begin{pmatrix} 0 & E^{-1}D \\ -E^{-1}D^T & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ -E^{-1}\phi(L) \end{pmatrix}, \; C = \begin{pmatrix} 0 & -\phi(L)^T \end{pmatrix}$$

Pour :

$$\begin{cases} \zeta = L = 1 \\ \phi(1) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{cases}$$

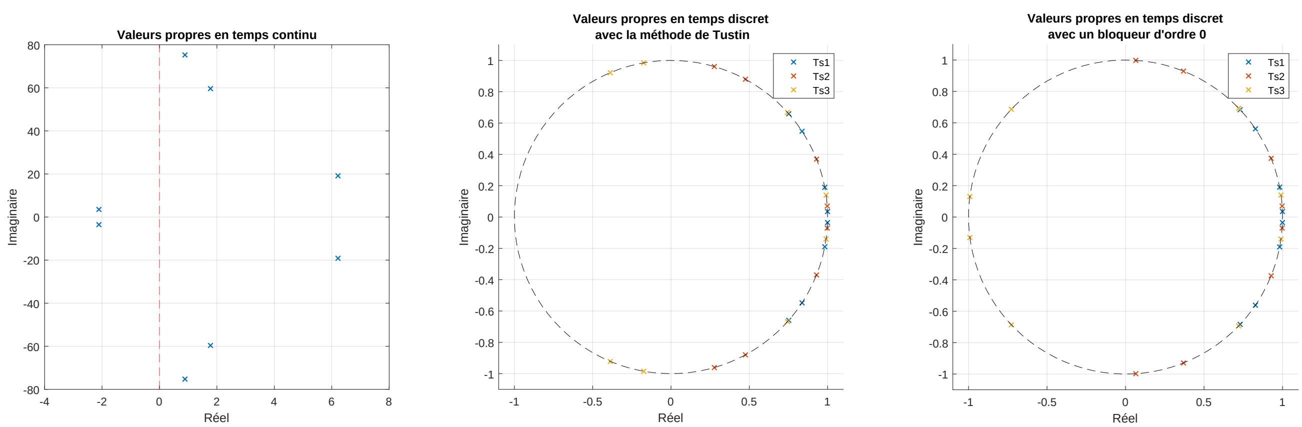
$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 114 & 6 & 12 & -2 \\ 0 & 0 & 0 & 0 & -54 & 6 & -2 & 4 \\ 0 & 0 & 0 & 0 & -1188 & -12 & -114 & 6 \\ 0 & 0 & 0 & 0 & -828 & -12 & -54 & 0 \\ 6 & 54 & 4 & -2 & 0 & 0 & 0 & 0 \\ -6 & -114 & -2 & 12 & 0 & 0 & 0 & 0 \\ -12 & -828 & -6 & 54 & 0 & 0 & 0 & 0 \\ -12 & -1188 & -6 & 114 & 0 & 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \\ -16 \\ -60 \\ -120 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

1.4 Question 4

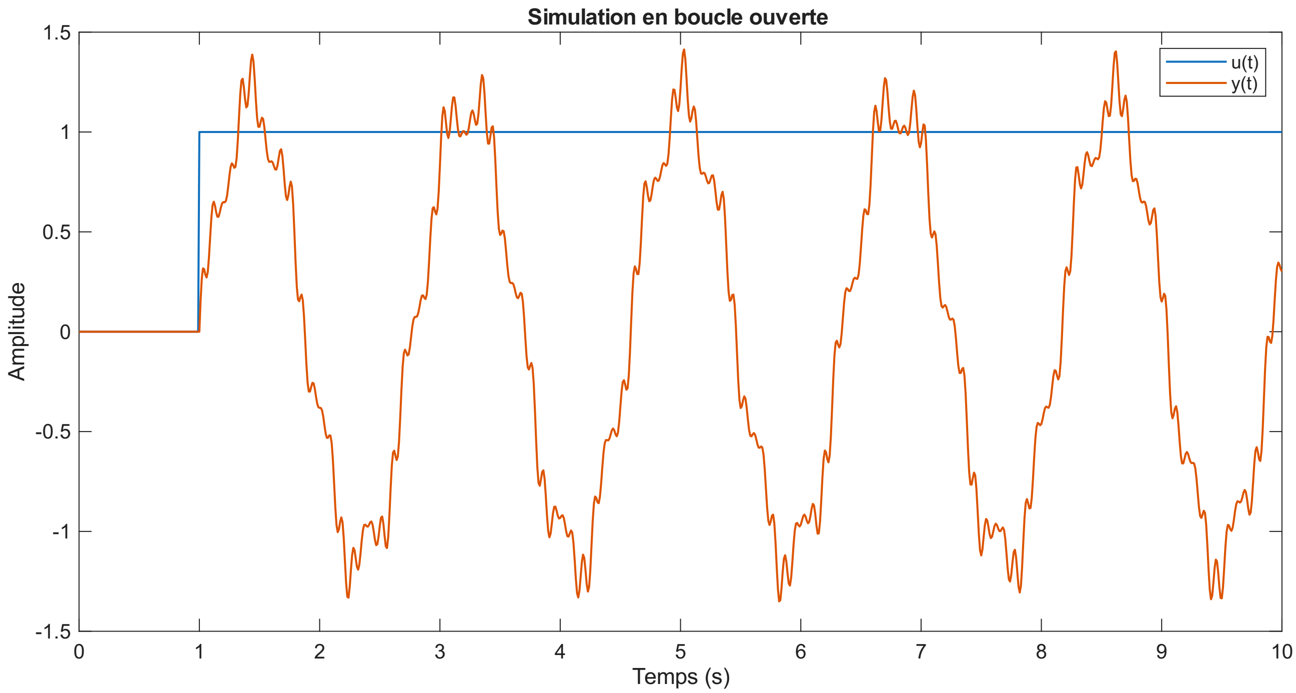
$$\omega(p,t) = C_\omega(p) x_d(t)$$

$$C_\omega(p) = \begin{bmatrix} \frac{p^2(2p^3-5p^2+10)}{20} & -\frac{p^4(2p-5)}{20} & \frac{p^3(3p^2-10p+10)}{60} & \frac{p^4(3p-5)}{60} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\phi(L)^T = \begin{bmatrix} 2p^3 - 3p^2 + 1 & 3p^2 - 2p^3 & p^3 - 2p^2 + p & p^3 - p^2 \end{bmatrix}$$



1.5 Question 5



$$y = -\phi(L)^T e_{2d} \qquad y = -e_2(L,t) \qquad e_2(\zeta,t) \approx \phi(\zeta)^T e_{2d}(t)$$

$$x_1 = \frac{\partial^2\omega}{\partial\zeta^2}(\zeta,t) \approx \phi(\zeta)^2 x_{1d} \qquad x_2(\zeta,t) = \rho(\zeta)\frac{\partial\omega}{\partial t}(\zeta,t) \approx \phi(\zeta)^T x_{2d}$$

$$\begin{bmatrix}\dot{x}_{1d}\\ \dot{x}_{1d}\end{bmatrix} = A\begin{bmatrix}x_{1d}\\ x_{2d}\end{bmatrix} + Bu \qquad y = C\begin{bmatrix}x_{1d}\\ x_{2d}\end{bmatrix}$$

1.6 Question 6

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}\frac{\partial^2\omega}{\partial\zeta^2}(\zeta,t)\\ \rho(\zeta)\frac{\partial\omega}{\partial t}(\zeta,t)\end{bmatrix} = \begin{bmatrix}\phi(\zeta)^T & 0000\\ 0000 & \phi(\zeta)^T\end{bmatrix}\begin{bmatrix}x_{1d}\\ x_{2d}\end{bmatrix}$$

$$\frac{\partial^2\omega}{\partial\zeta^2}(\zeta,t) = \underbrace{\begin{bmatrix}\phi(\zeta)^T & 0 & 0 & 0 & 0\end{bmatrix}}_{C.I.=0+0} x_d(t)$$

$$\frac{\partial\omega}{\partial\zeta}(\zeta,t) = \int\frac{\partial^2\omega}{\partial\zeta^2}(\zeta,t)d\zeta = x_d(t)\int\begin{bmatrix}\phi(\zeta)^T & 0 & 0 & 0 & 0\end{bmatrix}d\zeta + C_1(t)$$

$$\omega(\zeta,t) = \int\int\frac{\partial^2\omega}{\partial^2\zeta^2}(\zeta,t)d\zeta^2 = x_d(t)\int\int\begin{bmatrix}\phi(\zeta)^T & 0 & 0 & 0 & 0\end{bmatrix}d\zeta^2 + \underbrace{\int C_1(t)d\zeta + C_2(t)}_{\zeta C_1(t)+C_2(t)}$$

C_1 et C_2 ?????????????????

2 RETOUR DE SORTIE

2.1 Question 7

La sortie est donnée par $y(t) = C\, x_d(t)$, donc le système en boucle fermée s'écrit :

$$\begin{cases} \dot{x}_d(t) = (A - BCk)\, x_d(t) + BH\, w_c(L,t) \\ w(L,t) = C_w(L)\, x_d(t) \end{cases}$$

En régime permanent, $\dot{x}_d(t) = 0$, donc :

$$0 = (A - BCk)\, x_d + BH\, C_w(L)\, x_d \quad \Rightarrow \quad x_d = -(A - BCk)^{-1} BH w_c(L,t)$$

En multipliant l'expression par $C_w(L)$, on trouve :

$$w(L,t) = C_w(L)\, x_d = -C_w(L)(A - BCk)^{-1} BH w_c(L,t)$$

Pour assurer le suivi $w(L,t) = w_c(L,t)$, il faut :

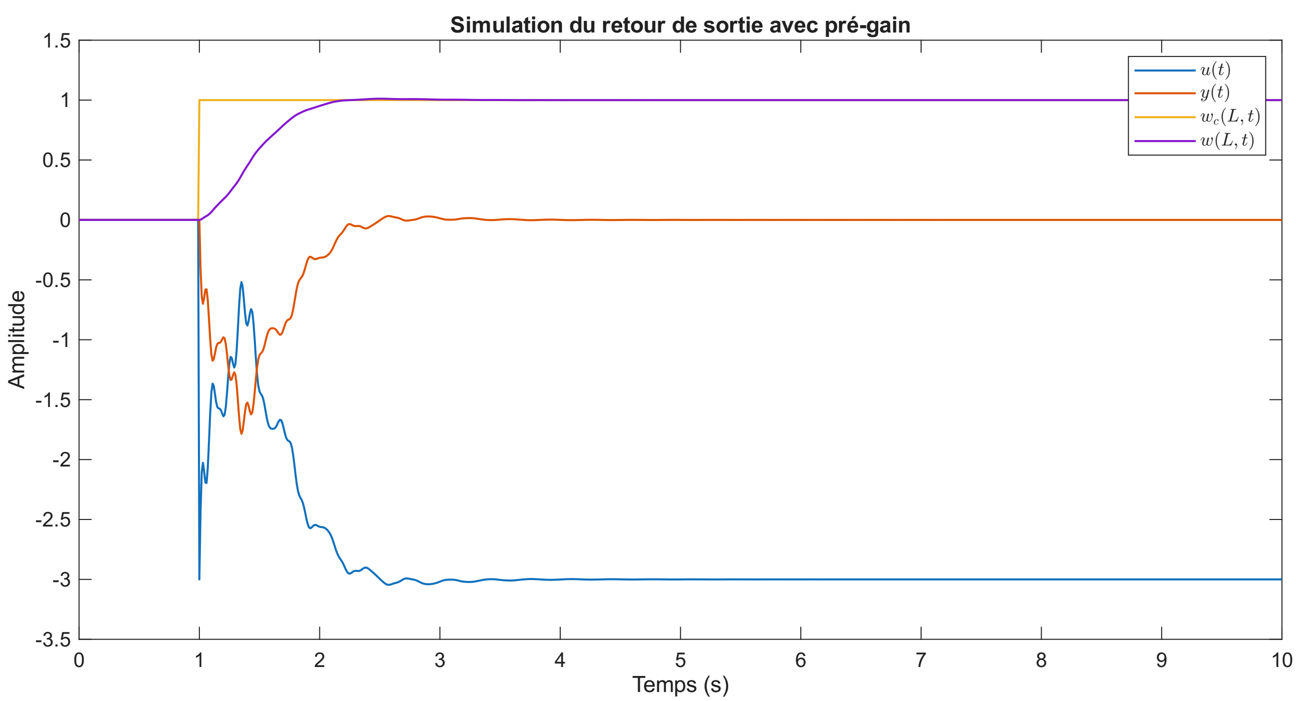
$$-C_w(L)(A - BCk)^{-1} BH = I$$

D'où :

$$\boxed{H = -\left(C_w(L)(A - BCk)^{-1} B\right)^{-1}}$$

Finalement, après calcul sous MATLAB, on obtient $H = -3$.

2.2 Question 8



2.3 Question 9

2.4 Question 10

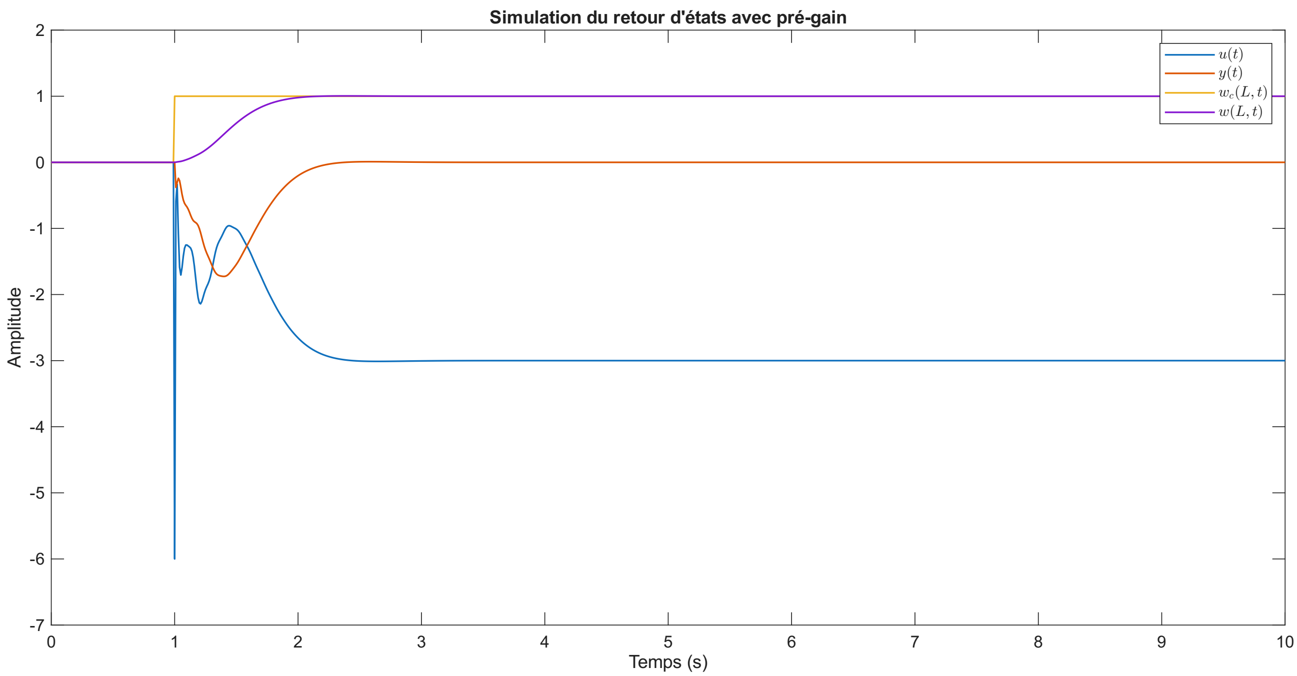
3 RETOUR D'ÉTAT

3.1 Question 11

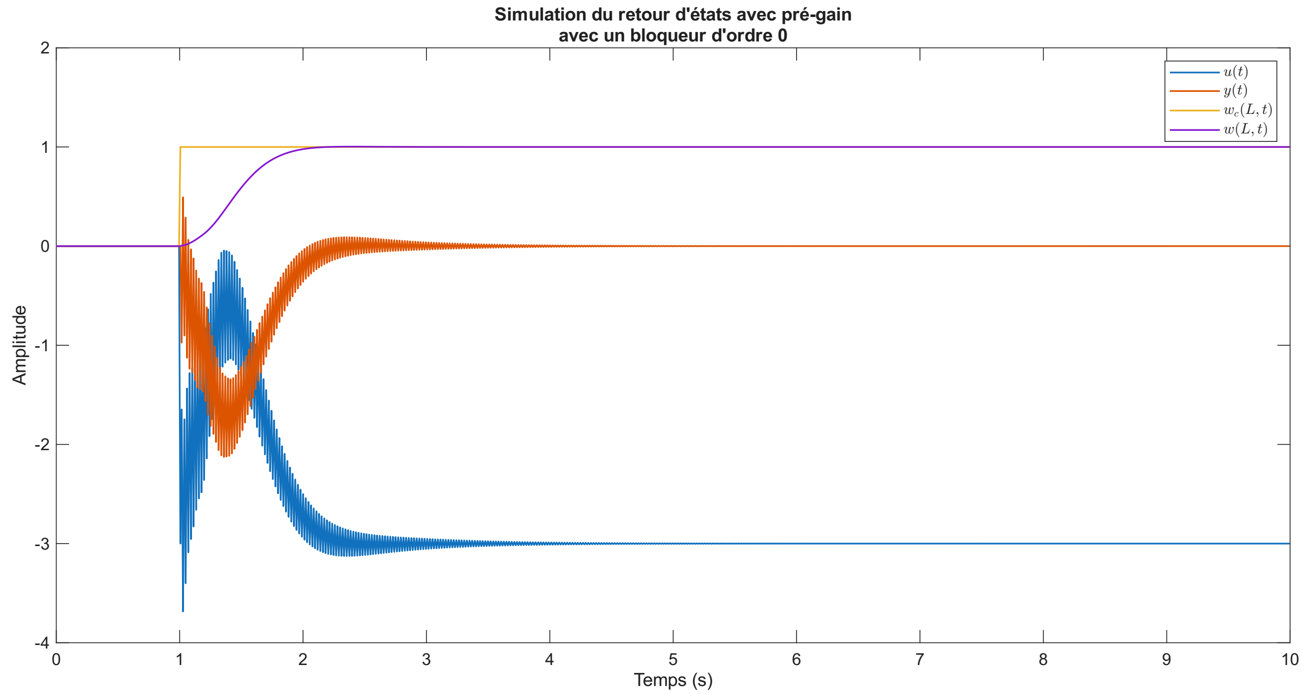
En utilisant la fonction `lqr()` de MATLAB avec $Q = I_8$ et $R = 1$, on trouve la matrice de gain K suivante :

$$K = \begin{pmatrix} 0.97 & -14.62 & 0.66 & 1.32 & 19.32 & 0.39 & 2.40 & -2.11 \end{pmatrix}$$

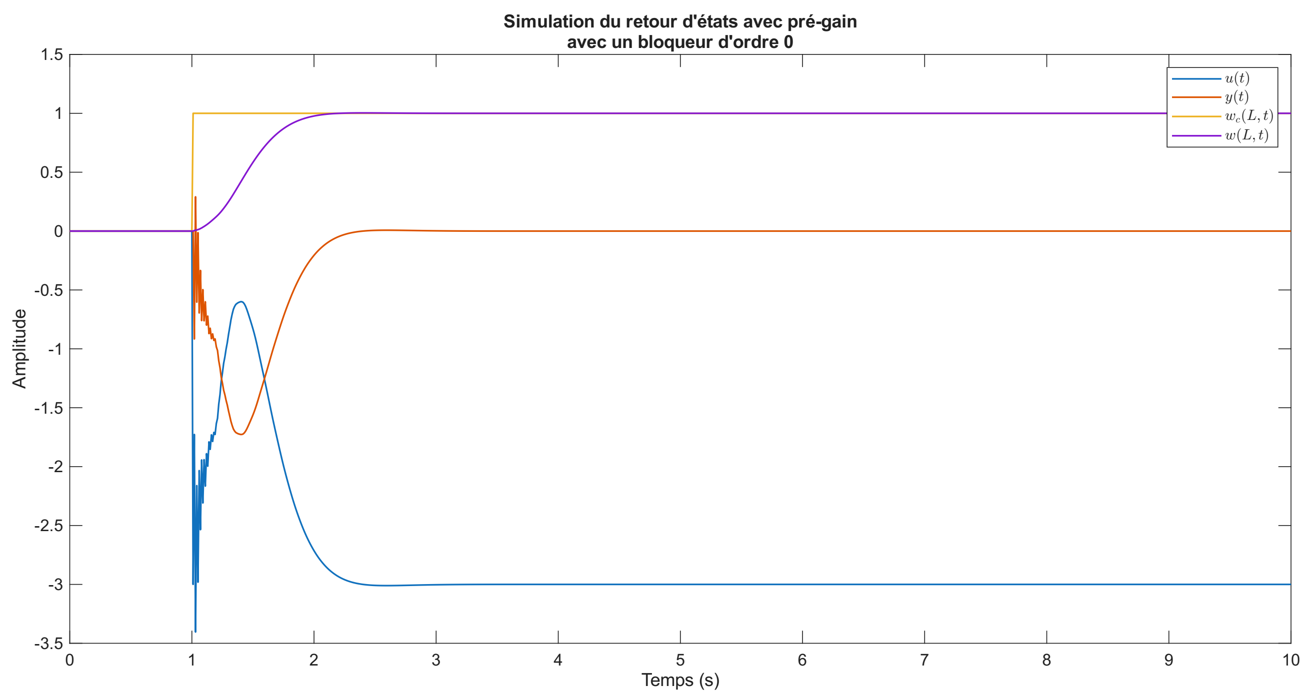
3.2 Question 12



3.3 Question 13



3.4 Question 14



4 REJECTION DE PERTURBATION

4.1 Question 15

On sait que :

$$F_{ext} = \int_0^L \phi(\zeta)q(\zeta,t)d\zeta \quad \Rightarrow \quad F_{ext} = q_0(t) \cdot \int_0^L \phi(\zeta)d\zeta \text{ car } q(\zeta,t) = q_0(t)$$

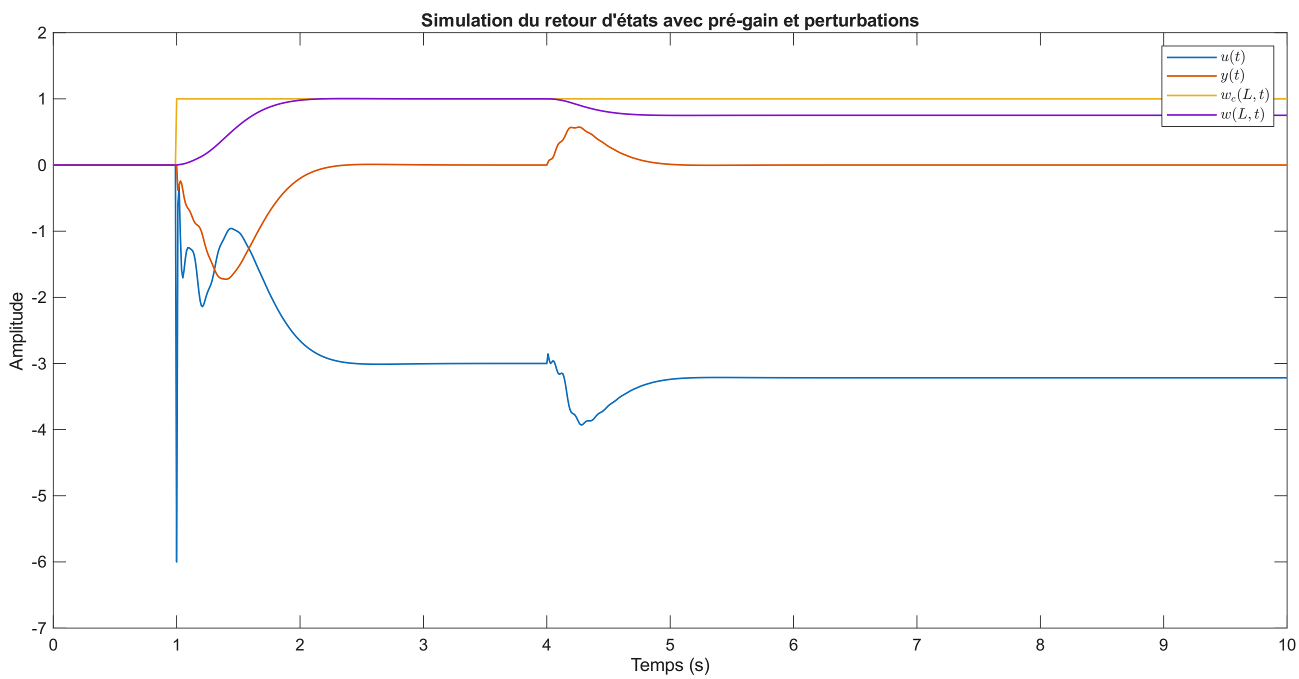
Or on a :

$$E\dot{x}_{2d} = -e_{1d}D^T - \phi(L)u(t) - F_{ext}$$

Donc, puisque $\dot{x}_d(t) = Ax_d(t) + Bu(t) + B_p q_0(t)$, on trouve :

$$B_p = \begin{pmatrix} 0_{4\times1} \\ -\int_0^L \phi(\zeta)d\zeta \end{pmatrix} = \begin{pmatrix} 0_{4\times1} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{12} \\ -\frac{1}{12} \end{pmatrix}$$

4.2 Question 16



Notre système ne permet pas de garantir une erreur nulle en régime permanent face à une perturbation constante.

4.3 Question 17

Nous souhaitons garder les mêmes valeurs propres que celles obtenues lors de la question 11 avec la LQR. La valeur propre supplémentaire doit être plus à droite pour permettre de rejeter la perturbation. On chosit $\lambda_9 = -2$. On trouve alors :

$$\lambda = \begin{pmatrix} -84.66 & -7.56 \pm 68.77i & -27.97 \pm 17.24i & -15.91 & -4.38 \pm 2.66i & -2 \end{pmatrix}^T$$

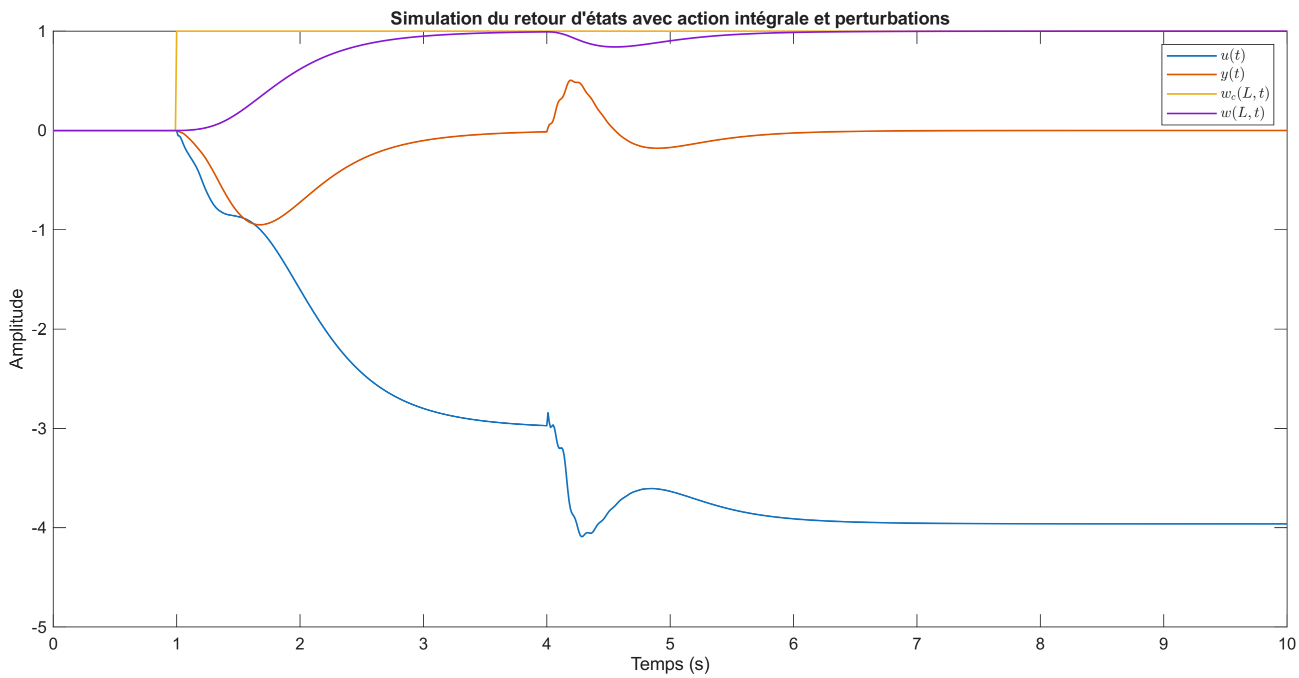
4.4 Question 18

En utilisant la fonction `place()` de MATLAB, on trouve la matrice de gain K suivante :

$$K_{aug} = \begin{pmatrix} K_1 & K_i \end{pmatrix} \quad \text{avec} \quad K_1 \in \mathbb{R}^{1\times 8},\ K_i \in \mathbb{R}$$

$$K_{aug} = \begin{pmatrix} -0.89 & -16.59 & 0.28 & 1.68 & 19.39 & -0.84 & 2.36 & -1.94 & 12.00 \end{pmatrix}$$

4.5 Question 19



En utilisant l'action intégrale, on parvient à rejeter la perturbation constante et à garantir une erreur nulle en régime permanent.

4.6 Question 20

5 ANNEXE

INSA TOULOUSE
135 avenue de Rangueil
31400 Toulouse

Tel : +33 (0)5 61 55 95 13
www.insa-toulouse.fr